Lecture 03
EN4562 Autonomous Systems
IMU Theory

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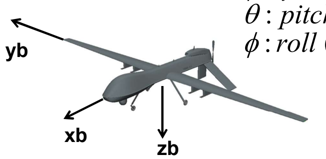
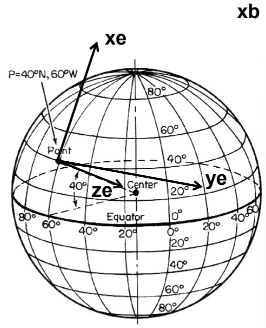
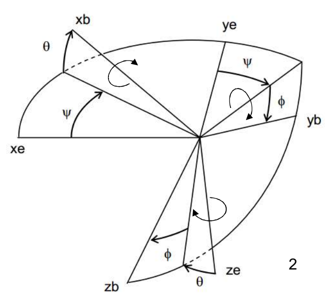
IMU Theory: Earth and Body Frames

- Xe : towards NORTH
- Ye : towards EAST
- Ze : into the centre of the Earth

Attitude
$\psi : yaw\,(z_B \text{ axis})$
$\theta : pitch\,(y_B \text{ axis})$
$\phi : roll\,(x_B \text{ axis})$







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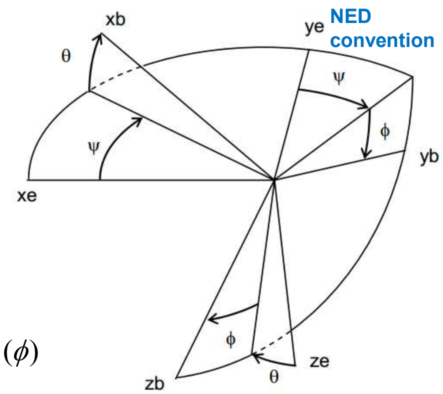
Inertial Measurement Sensors

- Measure orientation of {B} with respect to {E}
- Order: ψ (z_B axis-yaw), θ(y_B axis-pitch) and ϕ (x_B axis-roll)
- Orientation of {B} with respect to {E} is given by

$${}^{E}_{B}R = Rot_z(\psi)Rot_y(\theta)Rot_x(\phi)$$



$$\begin{array}{cccc} {}^{E}_{B}R = & \begin{bmatrix} \cos\theta\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \cos\theta\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix} \\ & r_{BxE} \qquad\qquad r_{ByE} \qquad\qquad r_{BzE} \end{array}$$

Rotation matrix, Direction Cosine Matrix (DCM)

$${}^{B}_{E}R = {}^{E}_{B}R^{-1} = {}^{E}_{B}R^{T}$$

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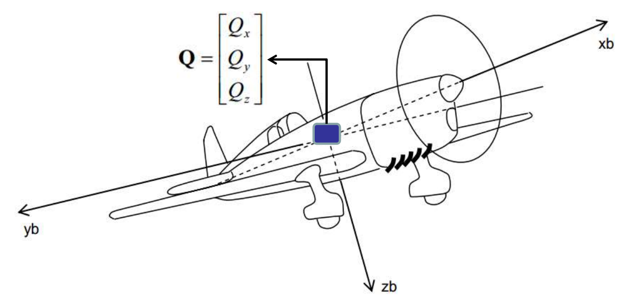
Body (sensor) Frame and Earth Frame

$$\mathbf{Q} = \begin{bmatrix} Q_x \\ Q_y \\ Q_z \end{bmatrix}$$



- Q_B = Sensor reading (acceleration, velocities ..)
- Q_E = Reading (vehicle motion) w.r.t. Earth (reference frame)

$${}^{E}Q = {}^{E}_{B}R\ {}^{B}Q$$

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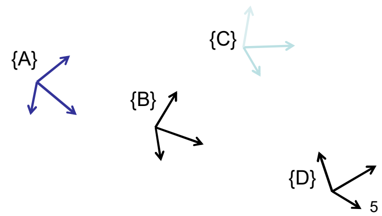
Order of Rotation is Important

- Eg:
  - A plane pitches by 90° and then rolls by 90°
    Plane is flying <u>vertically upwards</u>
  - A plane rolls by 90° and then pitches by 90°
    Plane is moving <u>horizontally</u>

- Compound Rotation (multiple frames, relative and reference)

$${}^{A}_{D}R = {}^{A}_{B}R\,{}^{B}_{C}R\,{}^{C}_{D}R$$



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Scalar and Vector Product

- Scalar (dot) Product

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{A}^{\mathbf{T}}\mathbf{B} = \begin{bmatrix} A_x & A_y & A_z \end{bmatrix}\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = A_xB_x + A_yB_y + A_zB_z$$

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cdot\cos(\theta_{AB})$$

- Vector (cross) Product
  - Components

$$(\mathbf{A}\times\mathbf{B})_x = A_yB_z - A_zB_y$$
$$(\mathbf{A}\times\mathbf{B})_y = A_zB_x - A_xB_z$$
$$(\mathbf{A}\times\mathbf{B})_z = A_xB_y - A_yB_x$$
$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$$

cross product operator

$$A\times B = \Omega_A B$$

$$A\times B = \begin{bmatrix} 1 & -A_z & A_y \\ A_z & 1 & -A_x \\ -A_y & A_x & 1 \end{bmatrix}\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$

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Small Orientation Change

- Let's consider the Rotation matrix of {B} w.r.t. {A} with yaw, pitch and roll

$${}^{E}_{B}R = \begin{array}{c} \begin{bmatrix} \cos\theta\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \cos\theta\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix} \\ \begin{array}{ccc} r_{BxE} \qquad\qquad & \qquad r_{ByE} \qquad\qquad & \qquad r_{BzE} \end{array} \end{array}$$

- Class Ex: Determine the incremental update rotation matrix due to a slight orientation change in YPR $(d\psi, d\theta, d\phi)^T$

Answer

$$\begin{pmatrix} 1 & -d\psi & d\theta \\ d\psi & 1 & -d\phi \\ -d\theta & d\phi & 1 \end{pmatrix}$$

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Incremental Orientation Update Using Gyro Sensors

- Class Ex: Now write the orientation update from R(t) to R(t+δ)t

$${}^{E}_{B}R(t+dt) = {}^{E}_{B}R(t)\begin{pmatrix} 1 & -d\psi & d\theta \\ d\psi & 1 & -d\phi \\ -d\theta & d\phi & 1 \end{pmatrix}$$

$$\text{where} \quad \begin{array}{rcl} d\psi &=& \omega_{Bz}dt \\ d\theta &=& \omega_{By}dt \\ d\phi &=& \omega_{Bx}dt \end{array} \quad \text{and } dt \approx \Delta t \leq 20\text{ms}$$

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Orientation Update with Gyro Readings

- About 50Hz update rate is required to keep the higher order terms negligible (we have used first order approximation)
- Update involves 27 multiplications and 18 additions, could be implemented on a processor that provides HW support for matrix multiplication
- A typical time step is around 20ms, during which an aircraft rotating at around 60° per second rotates approximately 0.020 radians, which translates to a maximum change in any of the R matrix coefficients of around 2%. Thus, the second order terms that are being ignored are on the order of 0.02%

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Preserving Orthogonality of R Matrix

- Three column vectors of **R** have to be orthogonal at all times.
- Numerical errors make column vectors non-orthogonal
- Numerical errors accumulate slowly, thus, need to be corrected periodically.
- **Renormalization**
  - **Correct the Matrix for orthogonality and normality**

$$ {}_B^E R = \begin{bmatrix} r_{BxEx} & r_{ByEx} & r_{BzEx} \\ r_{BxEy} & r_{ByEy} & r_{BzEy} \\ r_{BxEz} & r_{ByEz} & r_{BzEz} \end{bmatrix} $$

$$ {}_B^E R = \begin{bmatrix} r_{BxE} & r_{ByE} & r_{BzE} \end{bmatrix} $$

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Orientation Update with Gyro Readings

- Tests and simulations have shown that gyro based incremental orientation update yields accurate results with low drift, on the order of a few degrees per minute.
- Gyro drift, offset, and numerical errors accumulate slowly enough that it is possible to be ahead of it.

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Preserving Orthogonality of R Matrix

- Scalar product $(r_{\mathrm{BxE}} \cdot r_{\mathrm{ByE}})$ indicates how much the two unit vectors are non-orthogonal to each other
- The non-orthogonal error between x and y vectors of R

$$ r_{BxE} \bullet r_{ByE} = r_{BxE}^T r_{ByE} = \begin{bmatrix} r_{BxEx} & r_{BxEy} & r_{BxEz} \end{bmatrix} \begin{bmatrix} r_{ByEx} \\ r_{ByEy} \\ r_{ByEz} \end{bmatrix} $$

$$ e_{Rxy} = r_{BxEx} r_{ByEx} + r_{BxEy} r_{ByEy} + r_{BxEz} r_{ByEz} $$

If the error is 0, the two vectors are orthogonal

- The adjusted orthogonal vectors by cross-coupling

$$ {}^o r_{BxE} = r_{BxE} - 0.5 e_{Rxy} r_{ByE} $$

$$ {}^o r_{ByE} = r_{ByE} - 0.5 e_{Rxy} r_{BxE} $$

Homework: use two arbitrary vectors nearly orthogonal and implement this correction procedure

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Preserving Orthogonality of R Matrix

- Apportioning half error to each vector reduces the nonorthogonal error significantly under the condition that both vectors have nearly unit magnitude.
- The remaining vector ${}^{o}r_{BzE}$ of the R matrix is derived by cross product of the two orthogonal vectors as follows:

$$ {}^{o}r_{BzE} = {}^{o}r_{BxE} \times {}^{o}r_{ByE} $$

- After orthogonaity, renormalization is required to adjust vector magnitudes to unity (**Renormalization**)

Preserving Orthogonality of R Matrix

- Class Ex: Propose a method to adjust the magnitude of column vectors of R to unity

Answer: divide each element of each row by the square root of the sums of the squares of the elements in that row.

However, there is an easier way to do that, by recognizing that the magnitudes will never be much different than one, so we can use Taylor's expansion

Renormalization of R Matrix

- Using Taylor expansion, orthogonal vectors can be adjusted towards unit vectors as follows

$$ {}^{no}r_{BxE} = 0.5(3 - {}^{o}r_{BxE} \bullet {}^{o}r_{BxE})\,{}^{o}r_{BxE} $$
$$ {}^{no}r_{ByE} = 0.5(3 - {}^{o}r_{ByE} \bullet {}^{o}r_{ByE})\,{}^{o}r_{ByE} $$
$$ {}^{no}r_{BzE} = 0.5(3 - {}^{o}r_{BzE} \bullet {}^{o}r_{BzE})\,{}^{o}r_{BzE} $$

- This approximation doesn't have <u>square root</u> and <u>division</u> involved: can be computed quickly.
- Finally, renormalized, orthogonal Rotation Matrix is updated as follows:

$$ {}^{noE}_{B}R(t+dt) = \left[{}^{no}r_{BxE}(t+dt) \quad {}^{no}r_{ByE}(t+dt) \quad {}^{no}r_{BzE}(t+dt) \right] $$

Gyro Drift

- MEMS gyro sensors show a continuous drift at a rate of a few degrees per second. Thus, drift cancellation is critically important.
- An orientation reference (GPS? Accelerometer?, magnetometer?), that does not drift should be used to continuously compensate the reading for gyro drift.

Class Quiz: Guess what are those potential reference signals

- Magnetometers are useful for copters that hover.
- GPS will be useful for aircrafts.
- Accelerometers will provide a reference vector for the Z axis

Gyro Drift Correction

Method

- Cross product of a reference direction vector with the corresponding vector in R matrix (DCM), indicates the error. This error can be fed back through a correcting filter so that DCM vector could track the reference vector.
- Cross product tells the angle and axis of rotation needed to bring DCM vector to coincide with the Ref vector

Gyro Yaw Drift and GPS track Ref

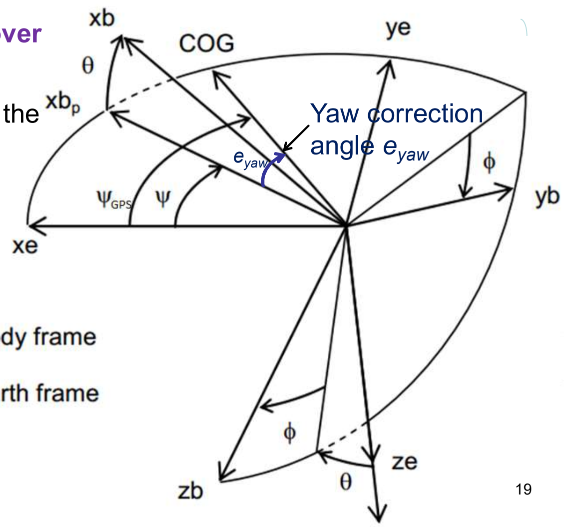
CoG : Course over Ground

ψ_{GPS} is given by the GPS CoG

$$\boldsymbol{r}_{CoG} = \begin{bmatrix} \cos\psi_{GPS} \\ \sin\psi_{GPS} \\ 0 \end{bmatrix}$$

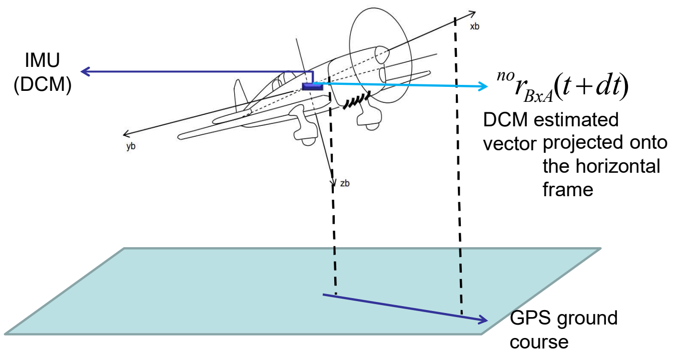
(xb yb zb) body frame

(xe ye ze) earth frame



Gyro Yaw Drift and GPS Track Ref

- GPS track provides yaw direction (heading) on the ground, thus, could be used as a Reference.



- For this reference to be useful, GPS must move. When the vehicle is not moving (hovering UAV). This method is useless

Gyro Yaw Drift and GPS track Ref

- Yaw Correction vector

$$\mathbf{e}_{yaw} = {}^{no}r_{BxA} \times r_{CoG}$$

Two vectors are on x-y plane of {E}

$$|{}^{no}r_{BxA}||r_{CoG}|\sin(\mathbf{e}_{yaw})\hat{\mathbf{e}}_{yaw} = \begin{bmatrix} {}^{no}r_{BxAx} \\ {}^{no}r_{BxAy} \\ {}^{no}r_{BxAz} \end{bmatrix} \times \begin{bmatrix} \cos\psi_{GPS} \\ \sin\psi_{GPS} \\ 0 \end{bmatrix}$$

unit↑ vector

$$\sin(e_{yaw})\hat{\mathbf{e}}_{yaw} = \begin{bmatrix} 0 & -{}^{no}r_{BxAz} & {}^{no}r_{BxAy} \\ {}^{no}r_{BxEz} & 0 & -{}^{no}r_{BxAx} \\ -{}^{no}r_{BxEy} & {}^{no}r_{BxA} & 0 \end{bmatrix} \begin{bmatrix} \cos\psi_{GPS} \\ \sin\psi_{GPS} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \approx 0 \\ \approx 0 \\ \sin(e_{yaw}) \end{bmatrix} = \begin{bmatrix} -{}^{no}r_{BxAz}\sin\psi_{GPS} \approx 0 \\ {}^{no}r_{BxAz}\cos\psi_{GPS} \approx 0 \\ -{}^{no}r_{BxAy}\cos\psi_{GPS} + {}^{no}r_{BxAx}\sin\psi_{GPS} \end{bmatrix}$$

Resulting vector is along z axis of {E}

Gyro Yaw Drift and GPS track Ref

- Yaw correction angle

$$\sin(e_{yaw}) = -{}^{no}r_{BxAy}\cos\psi_{GPS} + {}^{no}r_{BxAx}\cos\psi_{GPS}$$
$$e_{yaw} = \sin^{-1}(-{}^{no}r_{BxAy}\cos\psi_{GPS} + {}^{no}r_{BxAx}\cos\psi_{GPS})$$

Correction of Gyro Yaw Drift

- To correct yaw drift error, estimated yaw vector ${}^{no}r_{BxA}$ of ${}^{noE}_{B}\boldsymbol{R}$ has to be rotated about its own z-axis ${}^{no}r_{BzA}$ by the yaw error angle
- Correction can better be implemented as a PI filter which takes not only the instantaneous error angle, but any accumulated error in recent times and gradually reach a zero-error alignment of the heading estimate with the GPS ground course

$$e_{yaw} \longrightarrow \boxed{\text{PI}} \longrightarrow e^{*}_{yaw} \longrightarrow R_z(e^{*}_{yaw})$$

- The yaw drift corrected estimate is given by

$${}^{\psi noE}_{B}R = {}^{noE}_{B}R\ R_z(e^{*}_{yaw})$$

Slight rotation about Z axis of the body frame

Correction of Gyro Yaw Drift

- Set the proportional gain significantly high to reduce the error quickly.

Class Quiz: Under what conditions could this compensation be incorrect?

Answer: It is assumed that the vehicle is moving in the direction it is pointing (x_B). In aircrafts, this assumption can be violated due to cross-wind.

Gyro Roll-Pitch Drift Correction

- Using Accelerometers: single axis, three axes
- Accelerometer outputs pure gravity vector (direct orientation measurement) when the sensor(vehicle) is not accelerating. When the vehicle accelerates/decelerates it outputs gravity+acceleration as three components w.r.t. the body frame.
- Assumption: Vehicle does not accelerate/decelerate in forward direction. When it does, it last only briefly with one exception where the vehicle takes long turns that generate lasting lateral accelerations- centrifugal acceleration A_{cen}
- **Accelerometer output = Gravity + Centrifugal acceleration w.r.t Body (sensor) frame in NED**

$${}^{B}A_{sen} = {}^{B}g + {}^{B}A_{cen}$$

Gyro Roll-Pitch Drift Correction using Accelerometers

- With respect to body frame assuming zero error gyro o/p

  Note: Centrifugal acceleration = $v^2/r = r\omega^2 = \omega v$

Forward speed

$$
\begin{aligned}
{}^{B}A_{cen} &= {}^{B}\omega_{gyro} \times {}^{B}v \\
&= \begin{bmatrix} 0 & -{}^{B}\omega_{zgyro} & {}^{B}\omega_{ygyro} \\ {}^{B}\omega_{zgyro} & 0 & -{}^{B}\omega_{xgyro} \\ -{}^{B}\omega_{ygyro} & {}^{B}\omega_{xgyro} & 0 \end{bmatrix} \begin{bmatrix} {}^{B}v \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ {}^{B}\omega_{zgyro}\,{}^{B}v \\ -{}^{B}\omega_{ygyro}\,{}^{B}v \end{bmatrix}
\end{aligned}
$$

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Gravity Vector Estimate from Accelerometer Output

Gravity vector

$$
\begin{aligned}
{}^{B}g &= A_{sen} - {}^{B}\omega_{gyro} \times {}^{B}v \\
&= \begin{bmatrix} A_{xsen} \\ A_{ysen} \\ A_{zsen} \end{bmatrix} - \begin{bmatrix} 0 \\ {}^{B}\omega_{zgyro}\,{}^{B}v \\ -{}^{B}\omega_{ygyro}\,{}^{B}v \end{bmatrix}
\end{aligned}
$$

$$
\begin{bmatrix} {}^{B}g_x \\ {}^{B}g_y \\ {}^{B}g_z \end{bmatrix} = \begin{bmatrix} A_{xsen} \\ A_{ysen} - {}^{B}\omega_{zgyro}\,{}^{B}v \\ A_{zsen} + {}^{B}\omega_{ygyro}\,{}^{B}v \end{bmatrix}
$$

This is Gravity as seen from the body coordinate frame

$$
\sqrt{{}^{B}g_x^2 + {}^{B}g_y^2 + {}^{B}g_z^2} = g = 9.81
$$

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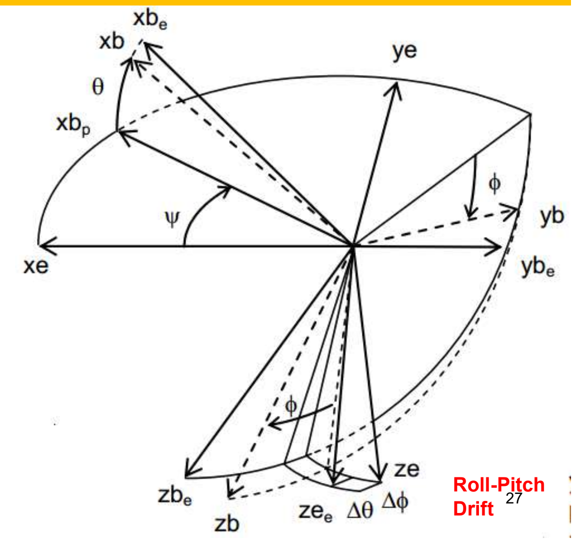
Gyro Roll-Pitch Drift Correction



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Roll and Pitch of {B} from Gravity Vector

- Roll angle using accelerometer readings $\varphi_A = \cos^{-1}\left(\dfrac{{}^{B}g_y}{g}\right)$
- Roll angle from the R matrix $\varphi_R = A\tan 2({}^{no}r_{ByEz}, {}^{no}r_{BzEz})$

  Ref Appendix
- Roll error (using accelerometer) $e_{\varphi Acc} = \varphi_A - \varphi_R$

Appendix

$$
{}^{noE}_{B}R = \begin{bmatrix} {}^{no}r_{BxEx} & {}^{no}r_{ByEx} & {}^{no}r_{BzEx} \\ {}^{no}r_{BxEy} & {}^{no}r_{ByEy} & {}^{no}r_{BzEy} \\ {}^{no}r_{BxEz} & \boxed{{}^{no}r_{ByEz} \quad {}^{no}r_{BzEz}} \end{bmatrix}
$$

$$
{}^{E}_{B}R = \begin{bmatrix} \cos\theta\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \cos\theta\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi \\ -\sin\theta & \boxed{\sin\phi\cos\theta \qquad \cos\phi\cos\theta} \end{bmatrix}
$$

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Pitch Error {B} from Gravity Vector

- Pitch angle from accelerometer readings $\theta_{Acc} = \cos^{-1}\left(\frac{{}^{B}g_x}{g}\right)$
- Pitch angle from the R matrix (Gyro)

$$\theta_R = A\tan 2\,(-{}^{no}r_{BxEz},\ {}^{no}r_{ByEz}/\sin\phi)$$

- Pitch angle error

Ref Appendix

$$e_{\theta Acc} = \theta_{Acc} - \theta_R$$

Appendix

$${}^{noE}_{B}R = \begin{bmatrix} {}^{no}r_{BxEx} & {}^{no}r_{ByEx} & {}^{no}r_{BzEx} \\ {}^{no}r_{BxEy} & {}^{no}r_{ByEy} & {}^{no}r_{BzEy} \\ \boxed{{}^{no}r_{BxEz} \quad {}^{no}r_{ByEz}} & & {}^{no}r_{BzEz} \end{bmatrix}$$

$${}^{E}_{B}R = \begin{bmatrix} \cos\theta\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \cos\theta\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi \\ \boxed{-\sin\theta \quad \sin\phi\cos\theta} & & \cos\phi\cos\theta \end{bmatrix}$$

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Attitude Estimation by Sensor Fusion

Gyro ${}^{noE}_{B}R \longrightarrow {}^{noE}_{B}R\ R_z(e^*_{\psi GPS})\ R_y(e^*_{\theta Acc})\ R_x(e^*_{\phi Acc})$

GPS $e_{\psi GPS} \rightarrow$ PI $\rightarrow e^*_{\psi GPS} \rightarrow R_z(e^*_{\psi GPS}) \rightarrow R_z(e^*_{\psi GPS})$

ACC

$e_{\theta Acc} \rightarrow$ PID $\rightarrow e^*_{\theta Acc} \rightarrow R_y(e^*_{\theta Acc}) \rightarrow R_y(e^*_{\theta Acc})$

$e_{\varphi Acc} \rightarrow$ PID $\rightarrow e^*_{\varphi Acc} \rightarrow R_x(e^*_{\varphi Acc}) \rightarrow R_x(e^*_{\varphi Acc})$

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Roll and Pitch Correction Using Accelerometer

- To correct pitch error, rotate estimated {B} by pitch error angle about ${}^{B}x$ axis

$e_{\theta Acc} \rightarrow$ PI $\rightarrow e^*_{\theta Acc} \rightarrow R_y(e^*_{\theta Acc})$

Slight rotation about Y axis $\quad {}^{\theta noE}_{B}R = {}^{noE}_{B}R\ R_y(e^*_{\theta Acc})$

- To correct roll error, rotate {B} by the error angle about ${}^{B}y$ axis

$e_{\varphi Acc} \rightarrow$ PI $\rightarrow e^*_{\varphi Acc} \rightarrow R_x(e^*_{\varphi Acc})$

Slight rotation about X axis $\quad {}^{\psi noE}_{B}R = {}^{noE}_{B}R\ R_x(e^*_{\phi Acc})$

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Class quiz: Could it be possible to use an accelerometer for direct orientation measurement?

Answer: No. The main reason is that they measure a combination of acceleration and gravity. If they measure only gravity, they would be perfect roll-pitch indicators. But they measure acceleration too, and that can cause trouble if used as roll-pitch indicators.

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